Dynamics of unimodal interval maps

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1 Discrete dynamical systems - basic definitions

Definition 1.1. ***Dynamical system*** *is a pair* (X, f)*, where* X *is a compact metric space and* $f\colon X \to X$ *is (piecewise) continuous. For* $n \in \mathbb{N}$ *denote by* $f^n := f \circ f \circ \ldots \circ f$ *(n times). The* ***forward orbit*** *of* $x \in X$ *is*

$$Orb(x) := \{x, f(x), f^2(x), f^3(x), \ldots\}.$$

The ω***-limit set of*** x *is a set of accumulation points of* $Orb(x)$*,* i.e.,

$$\omega(x, f) = \{y \in X : \textit{there exists a strictly increasing } (n_i)_{i\in\mathbb{N}},\ f^{n_i}(x) \to y \textit{ as } i \to \infty\}.$$

We say that x *is* ***periodic*** *if there exists* $n \in \mathbb{N}$ *such that* $f^n(x) = x$*. The smallest such* $n \in \mathbb{N}$ *is called a* ***prime period*** *of* x*. If* x *has period one, it is called a* ***fixed point****. If* x *is not periodic but there exists* $m \in \mathbb{N}$ *such that* $f^m(x)$ *is periodic, then* x *is called* ***preperiodic****. If* $x \in \omega(x)$*, then* x *is called* ***recurrent****.*

Definition 1.2. *Let* (X, f) *and* (Y, g) *be dynamical systems. We say that the systems (or maps* f*,* g*) are* ***topologically conjugate*** *if there exists a homeomorphism* $h\colon X \to Y$ *such that* $h \circ f = g \circ h$*. Such* h *is called a* ***topological conjugation*** *of* f *and* g*. If* h *can be taken at most continuous and surjective, systems are* ***topologically semi-conjugate*** *and* g *is called* ***a factor of*** f*. See Figure 1.*

Remark 1.3. *Conjugation* h *maps orbits of* X *to orbits of* Y*. Also* $h(\omega(x, f)) = \omega(h(x), g)$ *for all* $x \in X$*. Later we will see more invariants under topological conjugation. Often one thinks of conjugate systems as* dynamically equivalent*.*

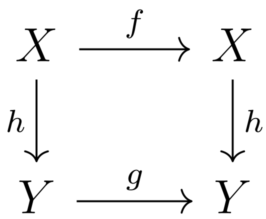


Figure 1: Topological (semi)-conjugation of f and g.

2 Unimodal maps

We will be studying dynamical properties of systems (I, f), where $I = [0, 1]$ and $f\colon I \to I$ is *unimodal*, defined below.

Definition 2.1. *We say that $f\colon I \to I$ is **unimodal** if*

(a) f is continuous,

(b) there exists a unique local maximum $c \in (0, 1)$, i.e., $f|_{[0,c)}$ is strictly increasing, $f|_{(c,1]}$ is strictly decreasing,

(c) $f(0) = f(1) = 0$.

Example 2.1. Typical families of unimodal maps are

(a) The **logistic family** $f_a(x) = ax(1 - x)$, $a \in [0, 4]$.

(b) The **tent family** $T_s(x) = \min\{sx, s(1 - x)\}$, $s \in [0, 2]$.

(c) The **sine family** $S_\alpha(x) = \alpha \sin(\pi x)$, $\alpha \in [0, 1]$.

See Figure 2. Note that f_a, T_s, S_α have a fixed point 0 for all parameters. Also, since

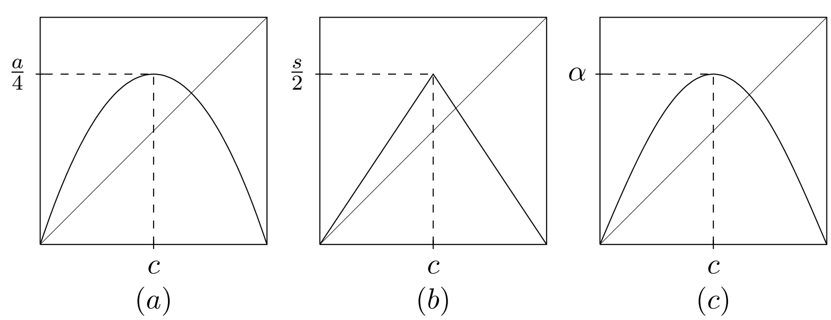


Figure 2: Graphs of (a) f_3, (b) $T_{1.5}$, (c) $S_{0.75}$.

$f_4(1/2) = T_2(1/2) = S_1(1/2) = 1$, and $f_4(1) = T_2(1) = S_1(1) = 0$, $c = 1/2$ is prefixed for f_4, T_2, S_1. For graphical representation of orbits see a **cobweb diagram** in Figure 3.

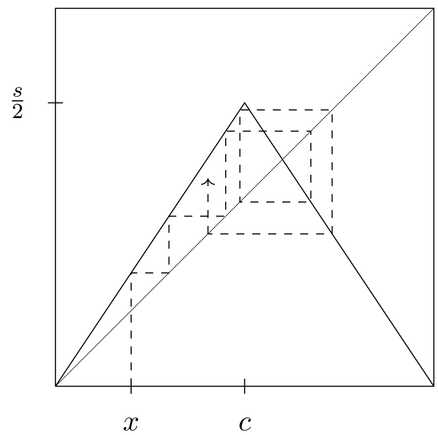


Figure 3: Cobweb diagram gives a graphical representation of an orbit. Take $x \in I$ and move up until you hit the graph of f. Then move horizontally until you hit the diagonal. That point if $(f(x), f(x))$. Continue vertically until you hit the graph and again horizontally until you hit the diagonal and obtain a point $(f^2(x), f^2(x))$. What is $\omega(x)$?

Remark 2.2. *Maps f_4, T_2 and S_1 are all topologically conjugate. Conjugation between f_4 and T_2 is given by $h(x) = \left(\sin(\frac{\pi x}{2})\right)^2$. Note that* e.g. *$f_2$ and T_1 are not conjugate since T_1 has an interval of fixed points while f_2 has two. In general there exist many parameters a for which f_a is conjugate to some tent map, but the conjugacies will not be smooth as in $a = 4$ case. As we will see later, every unimodal map is semi-conjugate to some tent map.*

3 Period doubling route to chaos

Definition 3.1. *Let x_0 be a periodic point of f with prime period n. We say that x_0 is*

(a) ***attracting*** *if $|\frac{d}{dx} f^n(x_0)| < 1$,*

(b) ***neutral*** *if $|\frac{d}{dx} f^n(x_0)| = 1$,*

(c) ***repelling*** *if $|\frac{d}{dx} f^n(x_0)| > 1$.*

Remark 3.2. *By the Mean Value Theorem, if x_0 is attracting, there exists an open set $U \ni x_0$ such that $\lim_{k\to\infty} f^{nk}(x) = x_0$ for all $x \in U$. If x_0 is repelling, there exists $U \ni x_0$ open such that for every $x_0 \neq x \in U$ there exists $k \in \mathbb{N}$ such that $f^{kn}(x) \notin U$.*

Example 3.1. (Period doubling cascade in the logistic family)
For $a < 1 =: a_1$ point 0 is attracting fixed point of f_a. Fixed point 0 is repelling for all $a > 1$ and neutral when $a = 1$.
If $1 < a < 3 := a_2$ then f_a has an attracting fixed point x_a which attracts all $x \in (0, 1)$.

That point becomes neutral when $a = 3$ and repelling when $a > 3$.
For $3 < a < 3.449489\ldots =: a_4$ there is an attracting period 2 cycle of f_a which becomes neutral in $a = a_4$ and repelling when $a > a_4$, creating a new attracting period 4 cycle. Further calculations show that the attracting 2^n-cycle becomes neutral at $a_{2^{n+1}}$ and repelling when $a > a_{2^{n+1}}$, creating an attracting period 2^{n+1} cycle. Parameters are approximately $a_8 = 3.54409\ldots$, $a_{16} = 3.56440726\ldots$, $a_{32} = 3.56875\ldots$, $a_{64} = 3.56969\ldots$, $\ldots$. The limit $\lim_{n\to\infty} a_{2^n} =: a_{feig} \approx 3.569945672\ldots$ is called a **Feigenbaum parameter**. Original results appeared in [14], [11].

Remark 3.3. *A qualitative change in the behavior of the system as in* e.g. $a = 1$, *where a single neutral periodic orbit appears and splits into stable and unstable periodic orbit is called a* saddle-node bifurcation. *A change as in* e.g. a_n, $n \geq 2$, *where a single attracting periodic orbit breaks into an attractive period* 2 *cycle is called a* period-doubling bifurcation.

Note that for $a^{2^n} < a < a^{2^{n+1}}$ a map f_a has a single periodic cycle of prime period 2^i for every $i = 0, \ldots, 2^n$ and the 2^n-cycle is attracting. Map $f_{a_{feig}}$ has periodic orbits of prime period 2^n for all $n \in \mathbb{N}_0$ and an attracting Cantor set (as we will later see). See Figure 4.

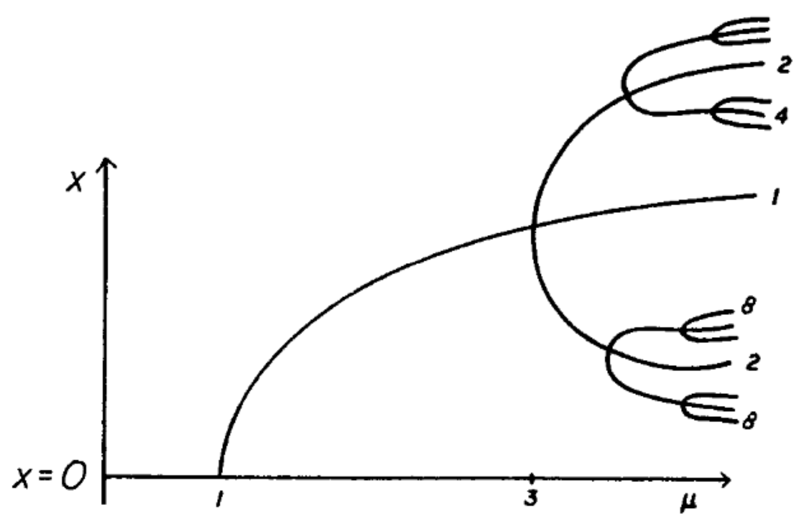


Figure 4: Period doubling in logistic family. Picture is taken from [17].

Numerics also indicate that $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{a_{n+2}-a_{n+1}} = \delta = 4.669201609\ldots$. What is fascinating is that δ is universal for families of unimodal maps which are smooth enough. Usually the smoothness assumption is the negativity of Schwarzian derivative, see the definition below. Constant δ is called the (first) **Feigenbaum constant**. The universality of δ was first noticed numerically (see [14],[11]) and explained using the renormalization theory, see *e.g.* [23] and the next section.

Definition 3.4. *Let $f\colon I \to I$ be continuous and at least three times differentiable. The* ***Schwarzian derivative*** *of f at x is*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

If $f'(x) = 0$ we define $Sf(x) = -\infty$.

*Unimodal map f for which $Sf(x) < 0$ for all $x \in I$ is called S****-unimodal****.*

Remark 3.5. *If $Sf < 0$, then f' cannot have a positive local minimum or negative local maximum. So between two successive extrema of f' there must be a critical point of f.*

Example 3.2. Every map in logistic family and sine family is S-unimodal. Tent map is not S-unimodal since it is not smooth at the critical point.

Theorem 3.6 (Singer, [30])**.** *If S-unimodal map has an attracting period orbit, then it attracts the critical point c. Thus, S-unimodal map has at most one attracting periodic orbit.*

Remark 3.7. *More generally, attracting orbits of interval maps with negative Schwarzian derivative attract a critical point or a boundary point. So, if $Sf < 0$ and f has n critical points, then the number of attracting periodic orbits is at most $n + 2$. Note that the definition of unimodal maps requires $f(0) = f(1) = 0$.*

4 Renormalization

Definition 4.1. *Unimodal map f is called* ***renormalizable*** *if there exists a closed interval $J \subset I$ and $n \geq 2$ such that*

(i) $f^n(J) \subset J$

(ii) J, $f(J)$, $\ldots f^{n-1}(J)$ *have disjoint interiors*

(iii) J *contains c in its interior.*

Interval J is called a ***restrictive interval of period*** *n and $f^n|_J\colon J \to J$ is called the* ***return map*** *or* ***renormalization*** *of f to J.*

Remark 4.2. *Note that $f^n|_J$ is again unimodal (possibly turned 'upside down',* i.e., *c is the minimum). Denote by $\varphi\colon J \to I$ an affine surjection such that $\varphi \circ f^n|_J \circ \varphi^{-1}\colon I \to I$ is unimodal (c is the maximum). Then $f \mapsto \mathcal{R}(f, J) = \varphi \circ f^n|_J \circ \varphi^{-1}$ is called a* ***renormalization operator****. Note that $\mathcal{R}(f, J)$ can again be renormalizable. In that case we say that f is* ***twice renormalizable*** *and analogously n-times renormalizable ($n = \infty$ is also allowed). See Figure 5.*

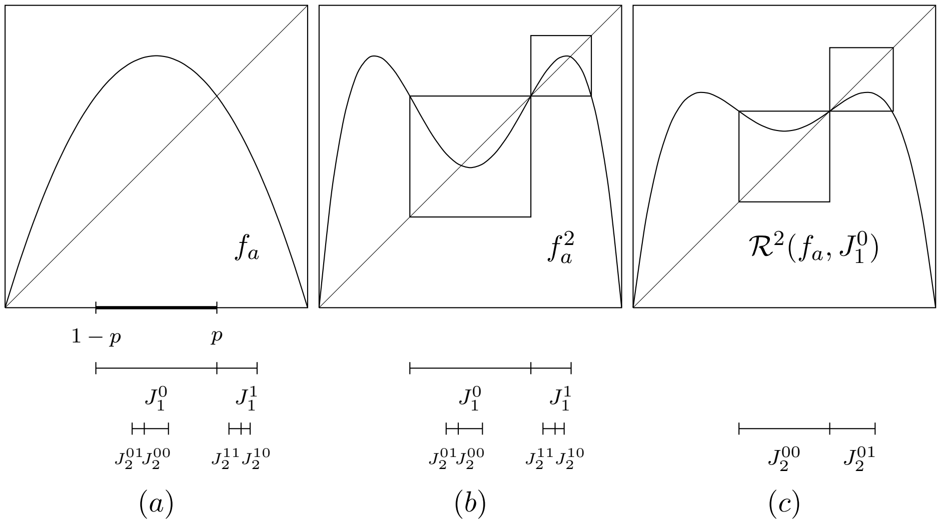


Figure 5: Figure (a) shows the graph of f_a for some $a_4 < a < a_8$. Note that it is renormalizable, with restrictive interval $J_1^0 = [1-p, p]$ of period 2, where $p > c$ is the fixed point of f_a. Denote by $J_1^1 = f_a(J_1^0) \subset [p, p']$, where $p' > p$ is such that $f_a^2(p') = p$. Then $f_a(J_1^1) \subset J_1^0$, thus $f_a^2(J_1^0) \subset J_1^0$. The graph of f_a^2 is given in (b). Note that $\mathcal{R}(f_a, J_1^0) = \varphi \circ f_a^2|_{J_1^0} \circ \varphi^{-1}$ is again unimodal and is again renormalizable, with restrictive interval J_2^{00} of period 2. Figure (c) shows the graph of $\mathcal{R}^2(f_a, J_1^0)$. Note that $\mathcal{R}(\mathcal{R}(f_a, J_1^0), J_2^{00})$ is no longer renormalizable. By $J_2^{i_1 i_2}$ we denote intervals such that $J_2^{i_1 i_2} \subset J_1^{i_1}$ and $f_a(J_2^{00}) = J_2^{10}$, $f_a(J_2^{10}) = J_2^{01}$, $f_a(J_2^{01}) = J_2^{11}$. Moreover, $f_a(J_2^{i_1 i_2})$ have disjoint interiors and $f_a^4(J_2^{i_1 i_2}) \subset J_2^{i_1 i_2}$ for all $i_1, i_2 \in \{0, 1\}$.

It can be shown that f_a is n-times renormalizable for $a_{2^n} \le a < a_{2^{n+1}}$, $n \in \mathbb{N}$. Thus $f = f_{a_{feig}}$ is n-times renormalizable for every $n \in \mathbb{N}$. It can be thought of as a fixed point of an operator $f \mapsto \varphi \circ f^2|_{J_n^{0\ldots 0}} \circ \varphi^{-1}$ (defined on a space of all infinitely renormalizable S-unimodal maps) where φ is an affine surjection as before and $J_n^{0\ldots 0}$ is a restrictive interval of period 2^n as in Figure 5. It turns out that this operator has a unique fixed point, with the single unstable eigenvalue equal to $\delta = 4.6692\ldots$ (the Feigenbaum constant). See [23].

For $i_1, \ldots, i_n \in \{0, 1\}$, $(i_1, \ldots, i_n) \neq (0, \ldots, 0)$ denote by

$$J_n^{i_1 \ldots i_n} = f^{i_1 + 2i_2 + \ldots + 2^{n-1} i_n}(J_n^{0\ldots 0}).$$

Note that $f(J_n^{i_1 \ldots i_n}) = f^{1 + i_1 + 2i_2 + \ldots + 2^{n-1} i_n}(J_n^{0\ldots 0}) = J_n^{i_1 \ldots i_n +_2 1}$, where $+_2$ denotes the binary addition "with carry", see the details below. Specially, $f(J_n^{1\ldots 1}) = f^{2^n}(J_n^{0\ldots 0}) \subset J_n^{0\ldots 0}$. Also note that $J_n^{i_1 \ldots i_n} \subset J_{n-1}^{i_1 \ldots i_{n-1}}$ for every $n \in \mathbb{N}$. We conclude that $\cap_n J_n^{i_1 \ldots i_n} =: C$ is a Cantor set and $\omega(x) \in C$ for every $x \in I$ which is not periodic (of period 2^n), specially $\omega(c) = C$. For every $x \in C$ write $x = x_1 x_2 \ldots \in \{0, 1\}^\infty$. Note that the action of f on C is given as binary "add one and carry":

$$f(x) = \begin{cases} (x_1+1)x_2x_3\ldots, & x_1 = 0, \\ 0(x_2+1)x_3\ldots & x_1x_2 = 10, \\ 00(x_3+1)x_4\ldots & x_1x_2x_3 = 110, \\ \ldots \end{cases}$$

Note that $f|_{\omega(c)}$ is one-to-one.

The construction easily generalizes to other infinitely renormalizable unimodal maps f.

Definition 4.3. *Let $\alpha = \langle q_1, q_2, \ldots\rangle$, $q_i \geq 2$. Define*

$$\Delta_\alpha = \{(x_1, x_2, \ldots), 0 \leq x_i < q_i\}.$$

Addition in Δ_α is $(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots)$, where $z_1 = x_1 + y_1 \pmod{q_1}$, $z_j = x_j + y_j + r_{j-1} \pmod{q_j}$, where $r_{j-1} = 0$ if $x_{j-1} + y_{j-1} + r_{j-1} < q_{j-1}$ and $r_{j-1} = 1$ otherwise (addition with carrying). The map $f_\alpha\colon \Delta_\alpha \to \Delta_\alpha$ given by

$$f_\alpha(x_1, x_2, \ldots) = (x_1, x_2, \ldots) + (1, 0, 0, \ldots)$$

*is called α-**adic adding machine**.*

Remark 4.4. *An action of $f_{a_{feig}}$ is the α-adic adding machine with $\alpha = \langle 2, 2, 2, \ldots\rangle$. It is called the **dyadic adding machine**.*

Remark 4.5. *If unimodal map f is infinitely renormalizable, then $\omega(c)$ is a Cantor set and $f|_{\omega(c)}$ is an α-adic adding machine (see* e.g. *[24]). The converse is not true, there exist non-renormalizable maps (specifically tent maps for dense set of parameters) with the above properties (see [4]).*

5 Chaos beyond the Feigenbaum parameter

Bifurcation diagram (see Figure 6) of logistic family (which is "universal" for S-unimodal maps, see Figure 7) indicates complicated behavior beyond the Feigenbaum parameter. However, we can observe intervals in parameter space with attracting periodic orbits (sometimes called "windows of stability"), which are again followed by a period doubling cascade resulting in an infinitely renormalizable map and chaos beyond. Windows of stability form a dense set in the parameter space (see [16], [22]). The set of parameters for which $\omega(x)$ equals an interval for Lebesgue almost every x is a Cantor set of positive Lebesgue measure (see [20], [3]).

Definition 5.1 (Devaney, [12])**.** *Let (X, f) be a dynamical system, where (X, d) is a metric space and $f\colon X \to X$ is continuous. We say that f is **Devaney-chaotic** on X if it is*

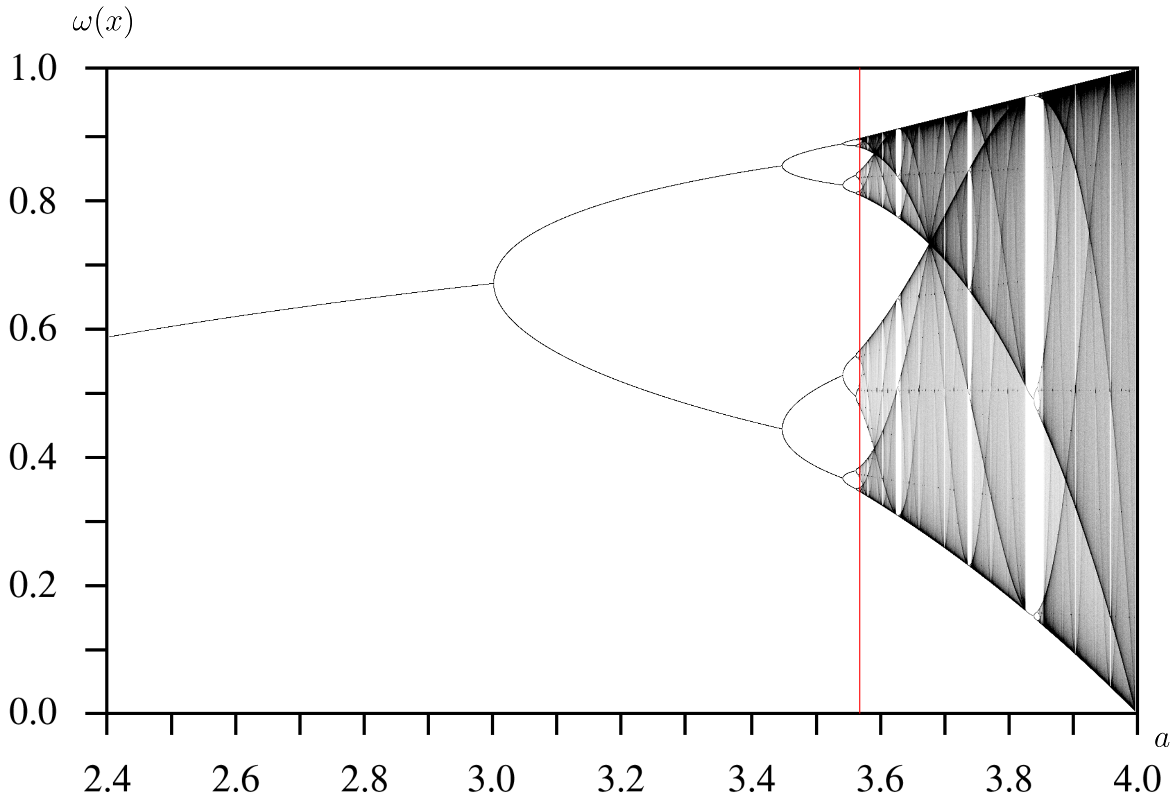


Figure 6: The bifurcation diagram of the logistic family. For every $a \in [2.4, 4]$ the set $\omega(c, f_a)$ is plotted in the vertical line $\{a\} \times I$. Red line denotes the Feigenbaum parameter.

(a) ***topologically transitive***, i.e., *for every open $U, V \subset X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$,*

(b) set of periodic points of f is dense in X,

(c) f is ***sensitive to initial conditions***, i.e., *there exists $\delta > 0$ such that for every $x \in X$ and $U \ni x$ open there exists $y \in U$ and $k \in \mathbb{N}$ such that $d(f^k(x), f^k(y)) > \delta$.*

Theorem 5.2 ([32])**.** *Continuous map $f : J \to J$ on a closed interval J is chaotic on J if and only if there exists a point $x \in J$ such that $Orb(x)$ is dense in J.*

Remark 5.3. *Maps $f : I \to I$ with attracting periodic orbits are obviously not chaotic on I. We will later see that if topological entropy of a unimodal f is positive (see Definition 6.1), then f is semi-conjugate to a tent map with the same entropy (see Theorem 11.1) and thus there exists $X \subset I$ such that $f|_X$ is chaotic on X. Specifically, topological entropy of f_a is positive whenever $a > a_{feig}$ (see 6.2) so there exists $X_a \subset I$ such that $f_a|_{X_a}$ is chaotic.*

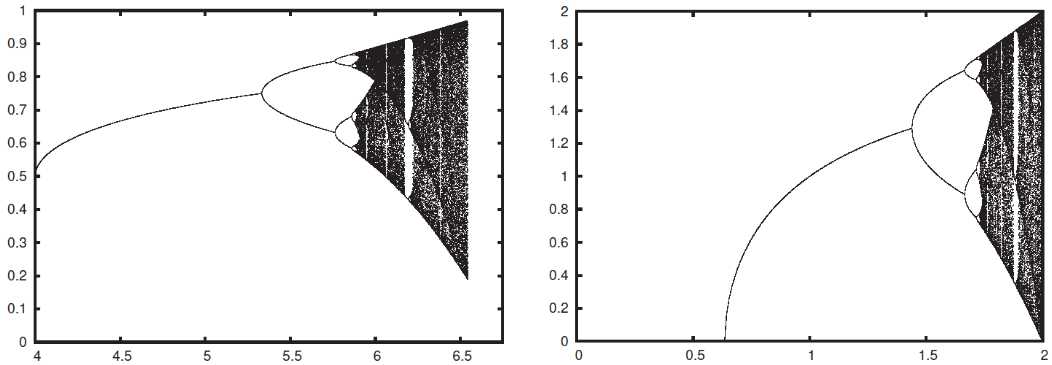


Figure 7: The bifurcation diagram of the family $ax^2(1-x)$ (on the left) and the sine family (on the right). Pictures are taken from [15].

6 Topological entropy

Definition 6.1. *Let (X, d) be a compact metric space, $f\colon X \to X$ a map (not necessary continuous). **Topological entropy** (Bowen-Dinaburg, see [5]) of f is*

$$h_{top}(f) = \lim_{\varepsilon \to 0} \limsup_{n} \frac{1}{n} \log s_{n,\varepsilon}(X, f),$$

where $s_{n,\varepsilon} = \inf\{|S| : S$ is (n,ε)-spanning$\}$. A set $S \subset X$ is (n,ε)-spanning if for every $x \in X$ there exists $y \in S$ such that $\max\{d(f^i(x), f^i(y)) : i \in \{0, \ldots, n\}\} \leq \varepsilon$.

Theorem 6.2 (Misiurewicz and Szlenk, [28])**.** *Let $f\colon I \to I$ be piecewise monotone. Then*

$$h_{top}(f) = \begin{cases} \max\{0, \lim_{n\to\infty} \frac{1}{n} \log l(f^n)\} \\ \max\{0, \lim_{n\to\infty} \frac{1}{n} \log Var_I(f^n)\} \\ \max\{0, \limsup_{n\to\infty} \frac{1}{n} \log Per_n(f)\} \\ \max\{0, \limsup_{n\to\infty} \frac{1}{n} \log \gamma_n(f)\}, f \text{ unimodal}, \end{cases}$$

*$l(f) = |\{J \subset I$ maximal$\colon f|_J$ monotone$\}|$ (**lap number** of f),*
*$Var_I(f) = \sup\{\sum_{i=1}^{n} |f(x_i) - f(x_{i+1})| : 0 = x_0 < x_1 < \ldots < x_n = 1\}$, (**variation**),*
$Per_n(f) = |\{$connected components of the set $\{x \in I : f^n(x) = x\}\}|$,
$\gamma_n(f) = |\{x \in f^{-n}(c) : f^k(x) \neq c, k = 0, \ldots, n-1\}|$.

Example 6.1. Note that $l(T_2^n) = 2^n$ and $l(f_4^n) = 2^n$ for all $n \in \mathbb{N}$ so $h_{top}(T_2) = h_{top}(f_4) = \log 2$.

Example 6.2. Since the variation of T_s^n grows as s^n, using the previous theorem, we can easily calculate $h_{top}(T_s) = \max\{0, \log s\}$, where T_s denotes the tent map with slope s. Note that $s \mapsto h_{top}(T_s)$ is continuous and monotone.

Remark 6.3. *In this remark we list some important facts about the topological entropy of interval maps.*

(i) If f and g are conjugate, then $h_{top}(f) = h_{top}(g)$. If f and g are semi-conjugate and g is a factor of f, then $h_{top}(f) \geq h_{top}(g)$.

For $s \geq 2$ an s-horseshoe of f is $J \subset I$ and a partition $\mathcal{D}$ of J into s subintervals such that $f(Cl(J_i)) \supset J$ for every $J_i \in \mathcal{D}$.

(ii) For $f\colon I \to I$ continuous $h_{top}(f) > 0$ if and only if f has a horseshoe, see [27]. Moreover, if f has an s-horseshoe, then $h_{top}(f) \geq \log(s)$.

(iii) For $f\colon I \to I$ continuous $h_{top}(f) > 0$ if and only if it has a cycle of period which is not a power of two, see [27].

Denote by $\mathcal{U}_1$ the space of all C^1-unimodal maps with C^1-topology and by $\mathcal{U}_0$ the space of C^0-unimodal maps with C^0-topology.

(iv) $\mathcal{U}_1 \ni f \mapsto h_{top}(f)$ is continuous, see [26].

(v) $\mathcal{U}_0 \ni f \mapsto h_{top}(f)$ is continuous at f_0 if $h_{top}(f_0) > 0$, see [27].

Monotonicity of entropy? Question of monotonicity of entropy in unimodal families is hard and still not quite understood. There exist families without the monotonicity of entropy, see [8]. It is known that $a \mapsto h_{top}(f_a)$ is monotone, see *e.g.* [26], [13] or [34] (see Figure 8). However, all the proofs use methods from complex analysis and question of "real" proof is still outstanding. The monotonicity of entropy for the sine family was recently proven, also by extending the maps to the complex plane, see [29].

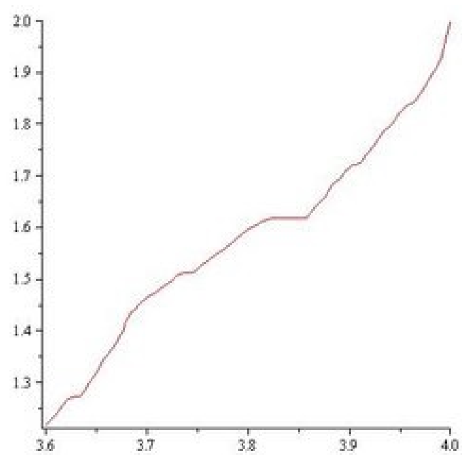


Figure 8: The devil's staircase $a \mapsto exp(h_{top}(f_a))$, where $f_a(x) = ax(1-x)$. The picture is taken from [19].

7 The Milnor-Thurston kneading theory

Most of the results in this section are based on the celebrated paper by Milnor and Thurston [26].

Let $f\colon I \to I$ be unimodal with critical point $c \in (0,1)$.

Definition 7.1. *Let $x \in I$. The **itinerary** of x is $i(x) = x_0x_1 \ldots \in \{0, C, 1\}^\infty$, where*

$$x_i = \begin{cases} 0, & f^i(x) < c \\ C, & f^i(x) = c \\ 1, & f^i(x) > c. \end{cases} \tag{1}$$

Equip $\{0, C, 1\}^\infty$ with the product topology (which is metrizable). Let $\sigma\colon \{0, C, 1\}^\infty \to \{0, C, 1\}^\infty$ be defined as $\sigma(x_0x_1x_2\ldots) = x_1x_2\ldots$. Then the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{\quad f \quad} & I \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \{0, C, 1\}^\infty & \xrightarrow{\ \sigma\ } & \{0, C, 1\}^\infty \end{array}$$

Note that i is not continuous exactly at preimages of c. Define $i(x^-) = \lim_{y \uparrow x} i(y)$, $i(x^+) = \lim_{y \downarrow x} i(y)$, where the limit is taken over $y \in I$ which are not preimages of c, i.e., *there is no $n \in \mathbb{N}$ such that $f^n(y) = c$.*

*The **kneading sequence** of f is a sequence $\nu_f = i(f(c))$. Note that if $C \in \nu_f$, the critical point c is periodic.*

Example 7.1. Take $f = T_2$, the full tent map. The itinerary of 0 is $i(0) = 000\ldots$. The itinerary of 1 is $i(1) = 100\ldots$. The itinerary of c is $i(c) = C100\ldots$ thus $\nu_f = 100\ldots$. Note that $i(c^-) = 0100\ldots$ and $i(c^+) = 1100\ldots$.

We want to introduce an ordering on itineraries which reflects an ordering on I.

Definition 7.2. *For $t_1 \ldots t_n \in \{0, C, 1\}^n$ denote by $\#_1(t_1 \ldots t_n)$ the number of symbols 1 in the word $t_1 \ldots t_n$. Let $s = s_0s_1 \ldots \in \{0, C, 1\}^\infty$. For $i \geq 0$ define*

$$\vartheta_i(s) = \begin{cases} +1, & \#_1(s_0 \ldots s_i) \text{ even,} \\ -1, & \#_1(s_0 \ldots s_i) \text{ odd.} \end{cases} \tag{2}$$

*Let $s = s_0s_1\ldots, t = t_0t_1 \ldots \in \{0, C, 1\}^\infty$ and let $i \in \mathbb{N}_0$ be the smallest such that $s_i \neq t_i$. We say that s is smaller than t in **parity-lexicographical ordering**, $s \prec t$, if either*

(a) $\vartheta_{i-1}(s) = +1$ and $s_i < t_i$, or
(b) $\vartheta_{i-1}(s) = -1$ and $s_i > t_i$ (where $0 < C < 1$).

Take $\vartheta_{-1}(s) = +1$ for completeness.

Example 7.2. Let $s = 111010\ldots$ and $t = 110111\ldots$. Then $\vartheta_1(s) = +1$, so since $1 > 0$, $s \succ t$ in the parity-lexicographical ordering.

Remark 7.3. *For $x \in I$ such that $f^k(x) \neq c$ for $0 \leq k \leq i$ (*i.e.*, C does not appear in the first $i+1$ symbols) we have $\vartheta_i(i(x)) = +1$ if f^{i+1} is locally increasing in x and $\vartheta_i(i(x)) = -1$ if it is locally decreasing. If $C \in i(x)$, then x is a critical point of f^{i+1}.*

We first discuss the injectivity of the map $x \mapsto i(x)$.

Lemma 7.4. *Let $x, y \in I$. Then*

$$x < y \implies i(x) \preceq i(y) \tag{3}$$

Remark 7.5. *The inequality in (3) cannot be improved in general. There can indeed exist points $x < y \in I$ such that $i(x) = i(y)$. In that case $f^n([x, y]) \not\ni c$ for every $n \in \mathbb{N}$ thus $i(z) = i(x) = i(y)$ for all $z \in [x, y]$. Intervals J such that $f^n|_J$ is a homeomorphism for every $n \in \mathbb{N}$ are called **homtervals**.*

Lemma 7.6 ([24])**.** *If J is a homterval, then*

- *(a) every point of J is attracted to an attracting periodic orbit, or*
- *(b) J is **wandering**,* i.e., *$J, f(J), f^2(J), \ldots$ are all disjoint and no point of J is attracted to an attracting periodic orbit.*

Theorem 7.7 (Guckenheimer, [18])**.** *S-unimodal maps do not have wandering intervals.*

Remark 7.8. *By Singer's theorem, if S-unimodal map has an attracting periodic orbit, then it must attract the critical point c (see Theorem 3.6). It is easy to see that in that case ν_f must be a periodic sequence. Thus, if ν_f is not periodic, $x \mapsto i(x)$ is one-to-one.*

Now we discuss the surjectivity of $x \mapsto i(x)$.

Remark 7.9. *Let $x \in I$. Note that by (3) it follows that $\sigma^k(i(x)) \preceq \nu_f$ for every $k \geq 0$. Next Lemma shows a partial converse.*

Lemma 7.10. *Assume f has no wandering intervals and assume ν_f is not periodic. Let $s = s_0 s_1 \ldots \in \{0, C, 1\}^\infty$. There exists $x \in I$ such that $i(x) = s$ if and only if either*

- *(a) $\sigma^k(s) \prec \nu_f$ for all $k \geq 0$, or*
- *(b) $\sigma^k(s) \preceq \nu_f$ for all $k \geq 0$ and if $\sigma^k(s) = \nu_f$ for a unique $k \geq 0$, then $s_{k-1} = C$.*

Corollary 7.11. *If f, g have no wandering intervals and have the same kneading sequence which is not periodic, then f and g are conjugate.*

Sketch of proof. For the construction of a conjugacy use the uniqueness of itineraries of f and g. □

Example 7.3. Every non-renormalizable logistic map with non-periodic kneading sequence (no attracting periodic orbit) is conjugate to some tent map. Later we will see that every logistic map beyond the Feigenbaum parameter is semi-conjugate to some tent map (Theorem 11.1).

Remark 7.12. *If ν_f is periodic, the conditions from Lemma7.10 become very technical, for details see Guckenheimer [18]. If ν_f is periodic, there can exist sequences satisfying (a) from the previous Lemma which cannot be realized as itineraries. For counterexample take* e.g. $T_{\frac{1+\sqrt{5}}{2}}$ *which has a kneading sequence $(10C)^\infty$ but no point has the itinerary $(101)^\infty \prec (10C)^\infty$.*

The following technical conditions will be needed later.

Definition 7.13. *Assume f is unimodal and $s \in \{0, C, 1\}^\infty$. We say that s is* dominated *by ν_f and write $s << \nu_f$ if for every $k \geq 0$:*

(a) $\sigma^k(s) \prec \nu_f$ and c is not periodic, or

(b) $\sigma^k(s) \prec (\nu_1 \dots \nu_n 0)^\infty$ and $\nu_f = (\nu_1 \dots \nu_n C)^\infty$ where $\#_1(\nu_1 \dots \nu_n)$ is even, or

(c) $\sigma^k(s) \prec (\nu_1 \dots \nu_n 1)^\infty$ and $\nu_f = (\nu_1 \dots \nu_n C)^\infty$ where $\#_1(\nu_1 \dots \nu_n)$ is odd.

Lemma 7.14 ([10], Theorem II.3.8)**.** *Let f be unimodal and $s \in \{0, C, 1\}^\infty$ such that $\sigma^k(s) << \nu_f$, then there is $x \in I$ such that $i(x) = s$.*

At the end of this section we state the necessary and sufficient conditions on a sequence to be a kneading sequence and discuss the universality of the logistic family.

Lemma 7.15 (Conditions on kneading sequences, [18])**.** *Let $s = s_0 s_1 \dots \in \{0, C, 1\}^\infty$ such that*

*(a) $\sigma^k(s) \preceq s$ for all $k \geq 0$ (we say that the sequence is **shift maximal**),*

(b) If $s_k = C$, then $s_{k+1+i} = s_i$ for $i \geq 0$.

Then there exists a S-unimodal map f such that $s = \nu_f$.

Lemma 7.16 (**Full family**, [18])**.** *Let $\{f_\nu : \nu \in [0, 1]\}$ be a continuous one-parameter family of S-unimodal maps such that $f_0(c) < c$, $f_1(c) = 1$. If $g \colon I \to I$ is a unimodal map, then there exists $\nu \in [0, 1]$ such that f_ν and g have the same kneading sequence.*

Remark 7.17. *Note that the logistic family satisfies the assumptions of the previous Lemma.*

8 Symbolics of renormalization

Assume f is renormalizable, *i.e.*, there exists $J \subset I$ and $n \geq 2$ such that $f^n(J) \subset J \ni c$ and $J, f(J), f^2(J), \ldots f^{n-1}(J)$ all have disjoint interiors (recall Definition 4.1). Then $g := f^n|_J$ is again unimodal with a kneading sequence ν_g. We want to describe ν_f with respect to ν_g. For details see [10].

Definition 8.1. *Let $A \in \{0,1\}^m$ and let $B = B_0B_1 \ldots \in \{0,C,1\}^\infty$. Define the* ****-product*** *as follows:*

$$A * B = \begin{cases} AB_0AB_1AB_2\ldots, & \text{if } \#_1(A) \text{ is even,} \\ A\hat{B}_0A\hat{B}_1A\hat{B}_2\ldots, & \text{if } \#_1(A) \text{ is odd, where } \hat{0}=1, \hat{1}=0, \hat{C}=C. \end{cases}$$

Remark 8.2. *Let f be renormalizable, with $J \subset I$ and $n \geq 2$ as in the definition. Note that $c \notin f^i(J)$ for all $i = 1, \ldots, n-1$. So we can assign $i(f^i(J)) \in \{0,1\}$ to each of those intervals, depending on their position with respect to c. Let $A = i(f^1(J)) \ldots i(f^{n-1}(J))$. Then $\nu_f = A * \nu_g$. See the Example below.*

Example 8.1. Let $f = f_a$ for some $a_8 < a < a_{16}$. Note that $\nu_f = (1011)^\infty = 1 * ((10)^\infty) = 1 * (1 * (1^\infty))$, since f is renormalizable with $n = 2$ and $f^2|_J$ is again renormalizable with $n = 2$. Denote by J' the restrictive interval of $f^2|_J$ which contains c. Then $f^4|_{J'}$ has an attracting fixed point with the itinerary $11\ldots$. See Figure 9.

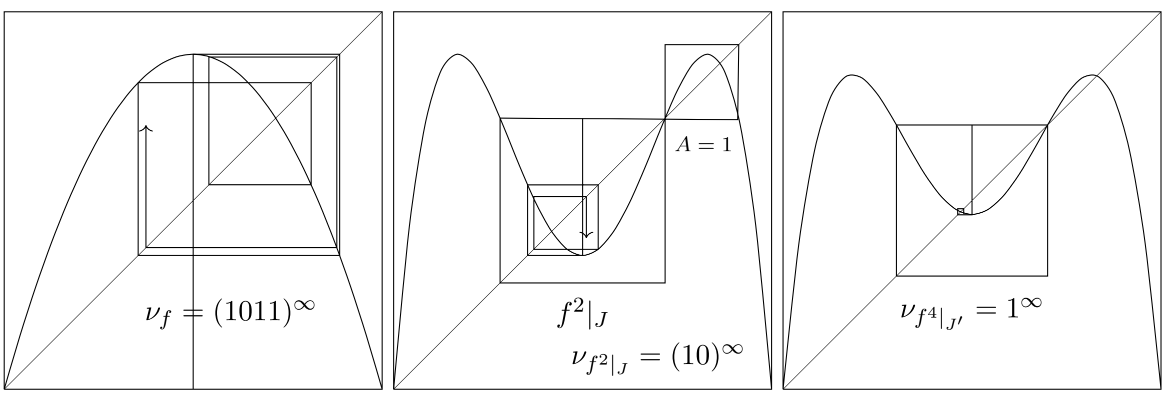


Figure 9: Cobweb plot for the map $f = f_a$, $a = 3.55$.

Example 8.2 (Kneading sequence of the Feigenbaum map f_{feig})**.** Denote the kneading sequence of f_a by ν_a.
For $a_1 < a \leq a_2$ there is an attracting fixed point and $\nu_a = 1^\infty$.
For $a_{2^n} < a \leq a_{2^{n+1}}$, $\nu_a = 1 * (1 * \ldots (1 * 1^\infty))$ (n times)
Since $a_{feig} = \lim_{n\to\infty} a_{2^n}$ we get

$$\nu_{feig} = 1^{*\infty} = 1011101010111011\ldots$$

This sequence can be constructed as follows:
Put 1 on every odd entry, leaving even spaces blank,

Put 0 on entries $2 + 4k$, $k \geq 0$, leaving other entries blank,
Put 1 on entries $4 + 8k$, $k \geq 0$, leaving other blank,
Put 0 on entries $8 + 16k$, $k \geq 0$, ...
This is an example of a Toeplitz sequence defined below. We note that this sequence can also be described as the fixed point of the substitution $1 \mapsto 10$, $0 \mapsto 11$, see *e.g.* [7].

Definition 8.3. *A sequence $(x_i)_{i\in\mathbb{N}} \in \{0,1\}^\infty$ is called **Toeplitz** if for every $i \in \mathbb{N}$ there exists $p_i > 1$ such that $x_i = x_{i+np_i}$ for every $n \in \mathbb{N}$.*

Remark 8.4. *Note that the kneading sequence of every infinitely renormalizable unimodal map is Toeplitz. Moreover, $f|_{\omega(c)}$ is conjugate to an adding machine if and only if ν_f is shift maximal, non-periodic Toeplitz sequence with* finite time containment property *(see [1]).*

9 The Sharkovsky theorem

In this section we will sketch the proof of the Sharkovsky theorem for unimodal interval maps using the symbolics. The theorem is valid in greater generality, see [31] for the original proof or standard textbook proof in *e.g.* [12, 27]. Symbolic proof was given in [10] and [17]. Partial results were obtained earlier, see [25].

Definition 9.1 (The Sharkovsky ordering)**.** *Define the following ordering on $\mathbb{N}$:*

$$\begin{aligned} 3 &\rhd 5 \rhd 7 \rhd \dots \\ &\rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \dots \\ &\dots \\ &\rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd \dots \\ &\dots \\ &\rhd \dots \rhd 2^n \rhd \dots \rhd 16 \rhd 8 \rhd 4 \rhd 2 \rhd 1. \end{aligned} \tag{4}$$

Theorem 9.2 (The Sharkovsky theorem)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. If f has a periodic point of prime period n, then f has a periodic point of prime period m for every $n \rhd m$.*

A symbolic proof of the Sharkovsky theorem in unimodal case relies on the order of periodic $0-1$ sequences in the parity lexicographical order. We will see that the order min-max sequences (defined below) reflects the Sharkovsky order.

Definition 9.3. *Define the set of periodic $0-1$ sequences which are periodic with prime period n and which are shift-maximal:*

$$S_n = \{s \in \{0,1\}^\infty : \sigma^n(s) = s, \sigma^k(s) \neq s \text{ for all } 1 < k < n, \sigma^k(s) \preceq s, \text{ for all } k \in \mathbb{N}\}.$$

*The minimal sequence in S_n (with respect to the parity lexicographical ordering) is denoted by P_n and called the **min-max sequence** of period n.*

Theorem 9.4 (See [10]). *If $s \lhd t \in \mathbb{N}$, then $P_s^\infty \prec P_t^\infty$.*
Moreover, $P_1 = 0$, $P_2 = 10$ and

$$P_n = \begin{cases} 101^{i-2}, & i \geq 3 \text{ is odd,} \\ P_{2^n} P_{2^{n-1}} P_{2^{n-1}}, & i = 2^{n+1} \\ P_{2^{n+1}} P_{2^n} \dots P_{2^n} (k-2 \text{ times}), & i = 2^n \cdot k,\ k \geq 3 \text{ is odd.} \end{cases}$$

Sketch of the proof of Sharkovsky's theorem when f is unimodal. Assume $p \rhd q$ and assume f has a periodic point x of (prime) period p. Then $i(x)$ is periodic sequence of period p. We can take $i(x)$ to be shift-maximal. The prime period of $i(x)$ is $p'|p$. If $p \neq 2^n$, then $p' \rhd p$ or $p' = p$. By the previous theorem, $P_{p'}^\infty \succeq P_p^\infty \succ P_q^\infty$. If $C \notin i(x)$, then we know that $\nu_f \succeq i(x) \succeq P_{p'}^\infty$ and thus $P_q^\infty \prec \nu_f$. From the relations above we conclude that $P_q^\infty << \nu_f$ so by Lemma 7.14 there exists a point of period q. If $C \in i(x)$, then $x = c$ and $\nu_f = (c_1 \dots c_{p-1} C)^\infty$. By taking $(c_1 \dots c_{p-1} 0)^\infty$ or $(c_1 \dots c_{p-1} 1)^\infty$ instead of $i(x)$ we derive the same conclusion as above. If $p = 2^n$, the conclusion follows from the following lemma. □

Lemma 9.5 ([10], Lemma II.3.2 and Lemma II.3.3). *If x is periodic of prime period p, then $i(x)$ is periodic of prime period p or $p/2$. If x is periodic of prime period p but $i(x)$ is periodic of prime period $p/2$, then there exists x' which is periodic of prime period $p/2$ and $i(x') = i(x)$.*

10 Symbolics of topological entropy

In this section we describe how symbolics can be used to compute the topological entropy. It was developed by Milnor and Thurston in [26]. In the same paper they prove the *monotonicity of entropy* in logistic family.

Let $f \colon I \to I$ be unimodal, with critical point $c \in (0,1)$ and denote by $\nu_f = i(f(c))$ the kneading sequence of f. Denote by $\Gamma_i = \{x \in I : f^i(x) = c, f^j(x) \neq c, j < i\}$ and by $\gamma_i = |\Gamma_i|$ for all $i \geq 0$. The power series

$$\gamma(t) = \sum_{i=0}^{\infty} \gamma_i t^i \in \mathbb{Z}[[t]]$$

is called the **cutting invariant** of f. Let r denote the radius of convergence of $\gamma(t)$. Then $1/r = \limsup_n (\gamma_n)^{\frac{1}{n}}$. Recall from Theorem 6.2 that $h_{top}(f) = \max\{0, \limsup_n \log(\gamma_n)^{\frac{1}{n}}\}$. So, if $h_{top}(f) > 0$, then $h_{top}(f) = \log(\frac{1}{r})$, where r is the radius of convergence of $\gamma(t)$.

The **lap invariant** of f is given by

$$l(t) = \sum_{i=0}^{\infty} l(f^{i+1}) t^i \in \mathbb{Z}[[t]],$$

where $l(f^i)$ denotes the number of laps of f^i as in Theorem 6.2.

Lemma 10.1. $l(t) = \frac{1+\gamma(t)}{1-t}$.

Recall the definition of itinerary and $i(x^{\pm})$ from (1) and (2). We abuse the notation and instead of writing $\vartheta_j(i(x))$ we write $\vartheta_j(x)$. The corresponding power series are

$$\vartheta(x,t) = \sum_{i=0}^{\infty} \vartheta_i(x)\, t^i, \quad \vartheta(x^+,t) = \sum_{i=0}^{\infty} \vartheta_i(x^+)\, t^i, \quad \vartheta(x^-,t) = \sum_{i=0}^{\infty} \vartheta_i(x^-)\, t^i \in \mathbb{Z}[[t]].$$

The **kneading invariant** $D(t)$ of f is defined as

$$D(t) = \vartheta(c^-, t) \in \mathbb{Z}[[t]].$$

Note that the radius of convergence of $D(t)$ is 1.

Theorem 10.2. *Let f be unimodal. Then $h_{top}(f) > 0$ if and only if $D(t)$ has a zero in $|t| < 1$. In that case, $h_{top}(f) = \log(\frac{1}{r})$, where r is the smallest zero of $D(t)$ in $[0,1)$.*

Sketch of proof. The proof follows directly from $D(t)\gamma(t) = \frac{1}{1-t}$ and $h_{top}(f) = \log(\frac{1}{r})$, where r is the radius of convergence of $\gamma(t)$. Use the extension of Abel's theorem (power series with positive coefficients and radius of convergence r has a singularity in r) to see that $\gamma(r)$ diverges and thus $D(r) = 0$. □

Using the kneading determinant and power series methods we can also calculate the number of periodic orbits of f. Assume f has finitely many periodic orbits of each period and denote by $Per_n(f)$ the number of fixed points of f^n for every $n \in \mathbb{N}$. The **Artin-Mazur zeta function** is defined as

$$\zeta(t) = \exp \sum_{n \geq 1} Per_n(f)\, \frac{t^n}{n}.$$

Theorem 10.3 (see [26])**.** *Let f be differentiable and such that all but finitely many periodic orbits are repelling (satisfied if* e.g. *$Sf < 0$). Then*

$$\frac{1}{\zeta(t)} = D(t) \prod_p \kappa_p(t),$$

where the product is taken over periodic orbits of f which are not repelling (and 0) and

$$\kappa_p(t) = \begin{cases} (1-t)^2, & \text{if } P = 0 \text{ and } 0 \text{ is attracting,} \\ 1-t, & \text{if } P = 0 \text{ and } 0 \text{ is not attracting,} \\ 1-t^k, & \text{if } P \text{ is attracting from one side only,} \\ 1-t^{2k}, & \text{if } P \text{ is attracting and } (f^k)'(P) < 0, \\ (1-t^k)^2, & \text{if } P \text{ is attracting and } (f^k)'(P) \geq 0. \end{cases}$$

Example 10.1. Take $f(x) = 3.84x(1-x)$. Then $f^{3i+1}(c) > c$, $f^{3i+1}(c) < c$, $f^{3i+2}(c) < c$ for all $i \geq 0$ (f belongs to the period 3 window). Then $\nu_f = (100)^\infty$ and $(\vartheta_i(c^-)) = (+1, -1, -1, -1, +1, +1, +1, -1, -1, -1, +1, +1, +1, -1, \ldots)$. So

$$D(t) = \vartheta(c^-, t) = 1 - t - t^2 - t^3 + t^4 + t^5 + t^6 - t^7 - \ldots = \frac{1 - t - t^2}{1 + t^3}.$$

Note that $\gamma(t) = \frac{1}{1-t} \frac{1+t^3}{1-t-t^2}$ and

$$l(t) = \frac{1}{1-t}(1 + \gamma(t)) = \frac{2}{1-t} \frac{1 - t + t^3}{1 - 2t + t^3} = 2 + 4t + 8t^2 + 16t^3 + 30t^4 + 54t^5 + 94t^6 + \ldots.$$

So, for example, f^7 has 94 laps.

Zeros of $D(t)$ are $\frac{-1 \pm \sqrt{5}}{2}$ so the smallest zero in $[0, 1)$ is $\frac{\sqrt{5}-1}{2}$. The topological entropy can be calculated as $h_{top}(f) = \log(\frac{2}{\sqrt{5}-1}) = \log \frac{\sqrt{5}+1}{2}$ (the golden mean!). From the previous theorem it follows that $\frac{1}{\zeta(t)} = \frac{1-t-t^2}{1+t^3}(1 - t)(1 - t^6)$, since 0 is repelling and there is an attracting period 3 orbit. So

$$\sum_{n \geq 1} Per_n(f)\, t^n = \frac{t\zeta'(t)}{\zeta(t)} = 2t + 4t^2 + 8t^3 + 8t^4 + 12t^5 + 22t^6 + 30t^7 + \ldots.$$

For example, it follows that f has 30 points of period 7. Two of them are fixed points of f and there are two period 7 cycles.

11 The piecewise linear model

In this section we construct, for a given unimodal map f of positive topological entropy, a semi-conjugacy to the tent map of the same entropy. The semi-conjugacy will collapse all intervals with considerably lower complexity than I to points. Specially, all homtervals will be collapsed.

Recall that we argued that f_4 and T_2 are conjugate, the conjugacy was given by $h(x) = \big(\sin(\frac{\pi x}{2})\big)^2$. The semi-conjugacies constructed in this section are typically not smooth.

Theorem 11.1. *Let $f\colon I \to I$ be unimodal and such that $h_{top}(f) > 0$. Denote by $s = \lim_n \frac{1}{n} l(f^n)$, where $l(f^n)$ is the lap number of f^n (s is usually called the* growth number *of f). Then f is semi-conjugate to T_s, the tent map with slope $s > 1$.*

Sketch of proof. Note that $h_{top}(f) = \log s$. Denote by $r = \frac{1}{s} < 1$. For $0 \leq a \leq b \leq 1$ define

$$\rho(a, b) = \lim_{t \uparrow r} \frac{\sum_{n=0}^{\infty} l(f^n|_{[a,b]})\, t^n}{\sum_{n=0}^{\infty} l(f^n)\, t^n}.$$

The map ρ measures the relative amount of laps of f^n on $[a, b]$. Note that if the number of laps of f^n on $[a, b]$ grows considerably slower than the total number of laps of f^n, then $\rho(a, b) = 0$. Define $h: I \to I$ as

$$h(s) = \rho(0, s)$$

and note that $h(0) = 0$, $h(1) = 1$, h is non-decreasing. Check the continuity of h using the fact $h_{top}(f) = \log s > 0$. Note that $l(f^n|_{[0,c]}) = l(f^n|_{[c,1]})$ for all $n \in \mathbb{N}$, so $h(c) = \frac{1}{2}$. For $x \in [0, c]$, $h(x) = r\rho(f(0), f(x)) = rh(f(x))$. Since $r = \frac{1}{s}$, we have $sh(x) = h(f(x))$. For $x \in [c, 1]$, $h(x) = h(c)+\rho(x, c) = h(c)+r\rho(f(c), f(x)) = h(c)+r(h(f(c))-h(f(x)))$, so $h(f(x)) = s(1 - h(x))$. Thus h is a semi-conjugacy of f and T_s. □

Remark 11.2. *The semi-conjugacy h can be given in a more useful form. Namely,*

$$h(x) = \frac{1}{2}(1 - (1 - r)\vartheta(x^-, r)).$$

Using this we plot the semi-conjugacy h between the logistic map $f_{3.84}$ (the attracting period 3 case) and the tent map $T_{\frac{1+\sqrt{5}}{2}}$. See Figure 10.

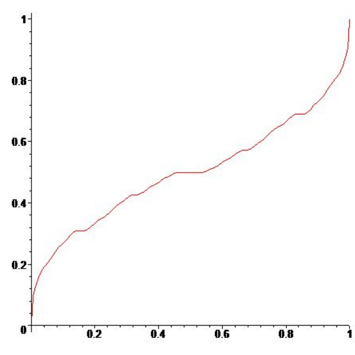


Figure 10: The function which semi-conjugates $f_{3.84}$ and $T_{\frac{1+\sqrt{5}}{2}}$. It collapses the restrictive interval J around c and all its images and preimages to points. We note that $I \setminus \cup_{k\in\mathbb{Z}} f^k(Int(J))$ is a Cantor set, denote it by C. Then $f_{3.84}|_C$ is conjugate to $T_{\frac{1+\sqrt{5}}{2}}$. The picture is taken from [19].

12 The dynamics of the tent family

In the previous section we showed that every unimodal map f such that $h_{top}(f) > 0$ is semi-conjugate to some tent map wit the same topological entropy. If $h_{top}(f) = 0$, then f is renormalizable. See the following Lemma.

Lemma 12.1 (See [33]). *If f is unimodal and $h_{top}(f) = 0$, then f is renormalizable.*

Sketch of proof. Restrictive interval containing c is *e.g.* $[p, \hat{p}]$ or $[\hat{p}, p]$, where p is the fixed point of f (not 0 or 1) closest to c and $\hat{p} \neq p$ is such that $f(\hat{p}) = p$. If there is no such p, we can take $p \in (0, 1)$ arbitrary. If such interval is not restrictive, there exists a horseshoe. □

If f does not have restrictive intervals or homtervals, then no proper subintervals of I have considerably slower growth rate of lap numbers compared to the entire map. We conclude that every unimodal map which is not renormalizable (or is renormalizable but only restrictive intervals are period 2^k of Feigenbaum type) and has no homtervals is topologically conjugate to some tent map with the same topological entropy. For details see *e.g.* [26] or [24]. In this section we study the dynamics of tent maps. Figure 11 shows the bifurcation diagram for the tent family.

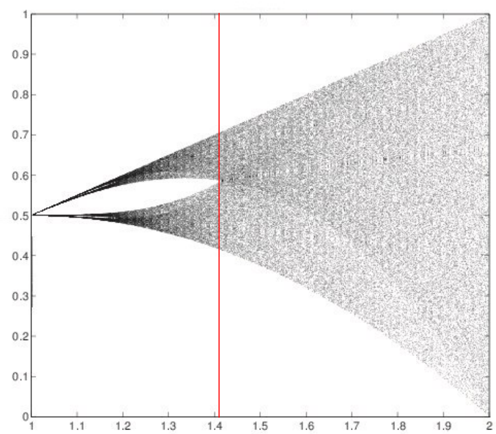


Figure 11: Bifurcation diagram of the tent family T_s for $s \in [1, 2]$. Finite renormalizations occur for $s \leq \sqrt{2}$.

We denote $T_s^n(c) = c_n$ for all $n \in \mathbb{N}$. When $s > 1$, let $p = \frac{s}{s+1}$ denote the fixed point of T_s in $(c, 1]$.

Remark 12.2. *Behavior of orbits in the tent family is as follows:*

1. *For $s < 1$, every point is attracted to the unique fixed point* 0.

2. *For $s = 1$, there is a continuum of fixed points and every other point is prefixed.*

3. *For $s > 1$, the interval $[c_2, c_1]$ is invariant (the **core**) and every point in $(0, c_2)$ is attracted to it.*

   (a) *If $1 < s \leq \sqrt{2}$, then T_s is renormalizable with restrictive interval $J = [c_2, p]$. Also $T_s(J) = [p, c_1]$ so every point in the core belongs either to J or its image. $T_s^2|_J$ is topologically conjugate to T_{s^2}. We conclude that if $\sqrt{2} < s^m \leq 2$ for some $m \geq 2$, then T_s is $m - 1$ times renormalizable. See Figure 12. No tent map is infinitely renormalizable.*

(b) *If $\sqrt{2} < s \le 2$, then T_s is not renormalizable. It is* ***locally eventually onto*** *on the core,* i.e., *for every open $U \subset [c_2, c_1]$ there exists $n \in \mathbb{N}$ such that $T_s^n(U) = [c_2, c_1]$. It follows that there is a dense orbit in $[c_2, c_1]$ and thus (see Theorem 5.2), $T_s|_{[c_2,c_1]}$ is Devaney chaotic.*

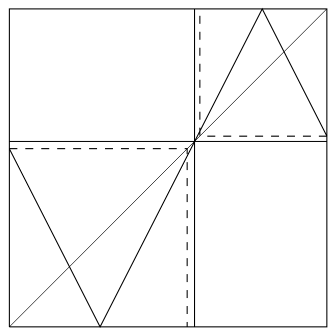


Figure 12: The graph of the map $T_s^2|_{[c_2,c_1]}$ where $s = 1.4 < \sqrt{2}$. Dashed lines denote the core of the renormalized map $T_s^2|_{[c_2,p]}$.

Remark 12.3. *Note that if $s > 1$ then for every interval U there exists $n \in \mathbb{N}$ such that $c \in T_s^n(U)$. It follows that if $x \ne y$, then $i(x) \ne i(y)$. Denote by ν_s the kneading sequence of T_s. If c is periodic, then $\nu_s = (\nu_1 \dots \nu_n C)^\infty$. We* modify the kneading sequence *according to the following rule: if $\#_1(\nu_1 \dots \nu_n)$ is even, define $\tilde{\nu}_s = (\nu_1 \dots \nu_n 0)^\infty$. If $\#_1(\nu_1 \dots \nu_n)$ is odd, define $\tilde{\nu}_s = (\nu_1 \dots \nu_n 1)^\infty$. If c is not periodic, define $\tilde{\nu}_s = \nu_s$. Modify the itineraries of points accordingly. That is, if $i(x) = x_0 x_1 \dots$ and $x_i = C$ for the smallest $i \ge 0$, then define $\tilde{i}(x) = x_0 \dots x_{i-1} C \tilde{\nu}_s$.*

Let $s = s_0 s_1 \ldots \in \{0, C, 1\}^\infty$ such that

1. *$\sigma(\tilde{\nu}_s) \preceq \sigma^k(s) \preceq \tilde{\nu}_s$, and*
2. *if $\sigma^k(s) = \tilde{\nu}_s$ for minimal $k \in \mathbb{N}$, then $s_{k-1} = C$.*

Then (see Lemma 7.14) there exists unique $x \in [c_2, c_1]$ such that $\tilde{i}(x) = s$.

13 Markov partitions and tent maps

Example 13.1. Take the map $T = T_s$ for $s = \frac{1+\sqrt{5}}{2}$. Then $\nu = (10C)^\infty$ and $\tilde{\nu} = (101)^\infty$. We know that sequence $s = s_0 s_1 \ldots \in \{0,1\}^\infty$ is realized as an itinerary of some $x \in [c_2, c_1]$ if and only if $(011)^\infty \prec \sigma^k(s) \prec (101)^\infty$. It holds if and only if $s_i = 0$ implies $s_{i+1} = 1$, *i.e.*, subword 00 is not allowed. Sequences satisfying that condition can be realized as infinite walks on the directed graph in Figure 13. Note that $T([c_2, c]) = [c, c_1]$, $T([c, c_1]) = [c_2, c] \cup [c, c_1]$ and $T|_{[c_2,c]}, T|_{[c,c_1]}$ are one to one.

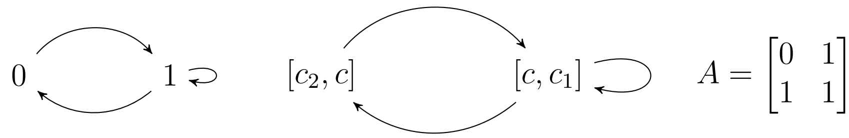


Figure 13: Markov graph for T_s with $s = \frac{1+\sqrt{5}}{2}$. Note that $\rho(A) = \frac{1+\sqrt{5}}{2}$.

Definition 13.1. *A **Markov partition** of a dynamical system (X, f) is a partition $\{X_i\}_{i=1}^n$ of X such that*

(a) $Int X_i \cap Int X_j = \emptyset$, if $i \neq j$,

(b) $f|_{X_i}$ is one to one for every i,

(c) If $f(Int X_i) \cap Int X_j \neq \emptyset$, then $f(X_i) \supset X_j$.

Remark 13.2. *Every Markov partition of (X, f) induces a directed **Markov graph** with vertices X_i, $i = 1, \dots, n$ and $X_i \to X_j$ if $f(X_i) \supset X_j$. Such graph induces the **transition matrix** $A = (a_{ij})_{i,j=1}^n$, where $a_{ij} = 0$ if $X_i \not\to X_j$ and $a_{ij} = 1$ if $X_i \to X_j$. See Figure 13.*

Lemma 13.3. *Assume (X, f) has a Markov partition with transition matrix A. Then the number of fixed points of f^n equals $\mathrm{tr} A^n$, the trace of A^n.*

Sketch of proof. The number of cycles of length n in a Markov graph equals $\mathrm{tr} A^n$. By the intermediate value theorem every cycle of length n in the Markov graph induces a periodic orbit of period n. The conclusion follows. □

Recall (see Theorem 6.2) that for piecewise linear map $f \colon I \to I$ we have $h_{top}(f) = \max\{0, \limsup_n \frac{1}{n} \log Per_n(f)\}$. If (I, f) has a Markov partition with the transition matrix A, it follows that $h_{top}(f) = \max\{0, \limsup_n \frac{1}{n} \log \mathrm{tr} A^n\}$. The following theorem tells us how to calculate this limit.

Definition 13.4. *A (directed) graph is **primitive** if there exists $m > 0$ such that for every pair of vertices v_i, v_j there is a (directed) path starting in v_i and ending in v_j of length $\leq m$. Equivalently, if there exists $m > 0$ such that $a_{ij}^m \neq 0$ for every $i, j = 1, \dots, n$, where $A^m = [a_{ij}^m]_{i,j=1}^n$.*

Theorem 13.5 (Perron-Frobenius)**.** *For primitive non-negative matrix A there exists a unique real eigenvalue $\lambda > 1$ such that $\lambda = \rho(A) = \max\{|\lambda_i| : \lambda_i$ is an eigenvalue of $A\}$.*

Remark 13.6. *If there is a unique maximal eigenvalue, then $\mathrm{tr} A^n \sim \rho(A)^n$ as $n \to \infty$. So, if A is primitive and non-negative, the Perron-Frobenius theorem says that $\lim_n \frac{1}{n} \log \mathrm{tr} A^n = \lim_n \frac{1}{n} \log \rho(A)^n = \log \rho(A) = \log \lambda$.*

Remark 13.7. *Every tent map with finite critical point induces a Markov partition of the core as in the example above. Note that the transition matrix of locally eventually onto tent map* $s > \sqrt{2}$ *will be primitive. From the Perron-Frobenius theorem it follows that* $s = \rho(A) = \lambda$.

Example 13.2. Take a periodic $0-1$ sequence which can be realized as the modified kneading sequence of some non-renormalizable tent map. That is every shift-maximal sequence of the form $(\nu_1 \dots \nu_n)^\infty$ where n is the prime period, $\#_1(\nu_1 \dots \nu_n)$ is even and it cannot be realized as a star product of two shift-maximal sequences. This follows from Lemma 7.15, Remark 8.2 and Theorem 11.1. For example, $(10111)^\infty$ is such a sequence. Since $c_2 < c < c_3 < c_4 < c_1$ (calculations are easily made symbolically), it follows that $\{[c_2, c], [c, c_3], [c_3, c_4], [c_4, c_1]\}$ forms a Markov partition of $[c_2, c_1]$. The corresponding diagram is given in Figure 14. The transition matrix is given by

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0. \end{bmatrix}$$

The largest eigenvalue of the transition matrix is $\lambda \approx 1.51288$. So $h_{top}(T_s) = \log s = \log \lambda$. Note that this also gives an effective way to calculate the slope of T_s with periodic critical point, given only its kneading sequence (rather than finding the roots of a polynomial).

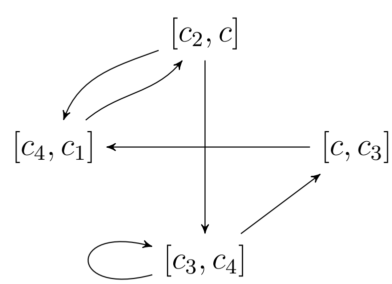
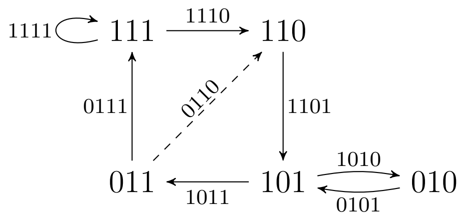


Figure 14: Markov graph for $\nu_s = (10111)^\infty$.

Remark 13.8. *We note that every tent map* T_s *induces a* shift space *and if* c *is periodic it is* subshift of finite type. *Every subshift of finite type induces a Markov graph. So given just a (finite) set of forbidden words, we can obtain a transition matrix and calculate an entropy of a system. In the example above, sequences* $s \in \{0,1\}^\infty$ *for which there is* $k \geq 0$ *such that* $\sigma^k(s) \succ (10111)^\infty$ *are exactly those which contain words* 00 *or* 0110. *So the set of forbidden words is* $\{00, 0110\}$. *From this we obtain a Markov graph in Figure 15 which also has the largest eigenvalue* $\lambda \approx 1.51288$. *For detail see standard textbooks in symbolic dynamics,* e.g. *[21].*



$$A=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 15: Markov graph for the subshift of finite type with forbidden words $\{00, 0110\}$. Nodes are admissible words of length 3. Arrows are labeled according to the action of the shift. Note that the arrow labeled with 0110 is not admissible (dashed). The transition matrix is given by A. It it primitive and its largest eigenvalue is $\lambda \approx 1.51288 = s$.

14 ω-limit sets of interval maps

In this section we discuss the topological properties of ω-limit sets of interval maps and classify them for tent maps. Recall that for (X, f), the ω-limit set of $x \in X$ is

$$\omega(x, f) = \{y \in X : \text{there exists a strictly increasing } (n_i)_{i\in\mathbb{N}}, f^{n_i}(x) \to y \text{ as } i \to \infty\}.$$

Example 14.1.

- If $f^n(x) = x$, then $\omega(x, f) = \{x, f(x), \dots, f^{n-1}(x)\}$.
- If $f = f_{feig}$ (see Section 3), then $\omega(c, f)$ is Cantor.
- Recall that *e.g.* T_2 has an orbit $Orb(x)$ which is dense in I. Then $\omega(x, T_2) = I$.

Remark 14.1. *For infinitely renormalizable unimodal maps, $\omega(c, f)$ is Cantor and $f|_{\omega(c)}$ is a homeomorphism (it is an adding machine). Such phenomenon can occur for non-renormalizable maps also,* e.g. *tent maps T_s with $s \in (\sqrt{2}, 2]$. Moreover, it turns out that such set of parameters s is dense in $[\sqrt{2}, 2]$ (see [7]). Symbolic classification of such parameters is a part of an ongoing research, see* e.g. *[2].*

Theorem 14.2 ([6])**.** *Let $f\colon I \to I$ be continuous. A set $M \subset I$ is an ω-limit set if and only if $M \neq \emptyset$ is nowhere dense in I or a finite union of closed intervals.*

Remark 14.3. *Let M be an ω-limit set which contains no intervals. Then it is nowhere dense (closure has empty interior) in I, so it contains no intervals, and thus it is totally disconnected (connected components are one point sets). Since M is closed in I, it is compact. Recall that a non-empty metric space is Cantor if it is perfect (there are no isolated points), compact and totally disconnected. If M is uncountable, then M can be uniquely expressed as $M = C \cup A$, where A is countable (can be $\emptyset$) and not closed, C is Cantor and $f(C) = C$ (for details see [7]). This gives the classification in the next proposition.*

Proposition 14.4. *Let $s \in (\sqrt{2}, 2]$ and x in the core $[c_2, c_1]$ of T_s. Then $\omega(x, T_s)$ is one of the following:*

(a) $[c_2, c_1]$ *(and set of such* x *has full Lebesgue measure),*

(b) totally disconnected

(i) finite; then x *is (pre)periodic.*

(ii) countable.

(iii) Cantor.

(iv) $C \cup A$*, where* A *is countable and not closed,* C *is Cantor and invariant.*

All cases can occur.

15 Attractors of unimodal interval maps

Definition 15.1. *Given* (X, f)*, a set* $A \subset X$ *is called a* ***topological attractor of*** f *if* $f(A) \subset A$ *and if its* ***basin*** $B(A) = \{x \in X : \omega(x) \subset A\}$ *satisfies:*

(a) $B(A)$ *is a residual set (*i.e., *its complement is a countable union of nowhere dense sets)*

(b) there is no proper subset of A *with this property.*

Example 15.1. Take a tent map T_s for $s > \sqrt{2}$. We know that there is $x \in [c_2, c_1]$ such that $\omega(x) = [c_2, c_1]$. So the topological attractor A is the whole $[c_2, c_1]$, with $B(A) = (0, 1)$. If $\sqrt{2} < s^m \leq 2$ for $m \geq 2$, then the topological attractor consists of $n = 2^{m-1}$ disjoint intervals $I_0, \dots I_{n-1}$ such that T_s maps I_j linearly onto $I_{j+1 \pmod{n}}$. Recall Figure 12.

Theorem 15.2 (Guckenheimer, [18])**.** *If* f *is* S*-unimodal, then* f *has at most one topological attractor and Lebesgue almost all points tend to the attractor.*

Theorem 15.3 ([18], [24])**.** *Let* $f \colon I \to I$ *be* S*-unimodal. Then the attractor of* f *is either:*

(a) a periodic orbit (periodic attracting orbit case)

(b) a finite union of intervals $I_1, \dots I_n$ *such that at least one of the intervals contains* c *and* $f^n|_{I_j}$ *is conjugate to a tent map (finite renormalization case),*

(c) a ***solenoidal*** *attractor* C*, where* $C = \omega(c)$ *is a Cantor set and* f *acts on* C *as an adding machine (infinite renormalization case).*

Definition 15.4. *For* (X, f) *a set* $A \subset X$ *is called a* ***metric attractor*** *if* $f(A) \subset A$*,* $|B(A)| > 0$ *and there is no proper subset with this property.*

Remark 15.5. *If f is S-unimodal and has an attracting periodic orbit, then that periodic orbit is both a topological and a metric attractor. Also, if f is infinitely renormalizable, the solenoidal attractor $\omega(c)$ is both a topological and a metric attractor. However, in the finite renormalization case it can happen that the topological and a metric attractor differ. There can exists a Cantor set C' in $I_1 \cup \ldots \cup I_n$ such that $B(C')$ has full Lebesgue measure but is not residual. Such set C' is called a **wild attractor**. Examples can be found in the family $f_\lambda(x) = \lambda(1 - |2x-1|^l)$ for large $l \in \mathbb{N}$. See [9]. There are no wild attractors in the logistic family.*

References

[1] L. Alvin, *Toeplitz kneading sequences and adding machines*, DCDS-A 33 (2013), 3277–3287.

[2] L. Alvin, *A characterization of unimodal maps with embedded adding machines*, preprint February 2017.

[3] M. Benedicks, L. Carleson, On iterations of $x \mapsto 1 - ax^2$ on $(-1, 1)$, Ann. Math. 122 (1985) 1–25.

[4] L. Block, J. Keesling, and M. Misiurewicz, *Strange adding machines*, Ergod. Th. & Dynam. Sys. 26 (2006), 673-–682.

[5] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 181 (1973), 509–510.

[6] A. M. Bruckner, J. Smital, *The structure of ω-limit sets for continuous maps of the interval*, Math. Bohem. 117 (1992), no. 1, 42–47.

[7] K. M. Brucks, H. Bruin, *Topics from one-dimensional dynamics*, London Mathematical Society, Student Texts 62 Cambridge University Press 2004.

[8] H. Bruin, *Non-monotonicity of entropy of interval maps*, Phys. Lett. A 202 (1995), no. 5-6, 359–362.

[9] H. Bruin, G. Keller, T. Nowicki, and S. van Strien, *Wild Cantor attractors exist*, Ann. of Math. (2) 143 (1996), no. 1, 97–130.

[10] P. Collet, J.-P. Eckmann, *Iterated maps of the interval as dynamical systems*, Birkhäuser, Boston 1980.

[11] P. Coullet, C. Tresser, *Itération d'endomorphismes et groupe de renormalisation*, C.R. Acad. Sci. Paris Sér. A-B 287 (1978), no.7, A577–A580.

[12] R. Devaney, *An introduction to chaotic dynamical systems*, Benjamin Cummings 1986.

[13] A. Douady, *Topological entropy of unimodal maps: Monotonicity for quadratic polynomials*, Real and Complex Dynamical Systems, B. Branner and P. Hjorth, Eds. Dordrecht: Kluwer (1995), 65–87.

[14] M. J. Feigenbaum, *Quantitative universality for a class of nonlinear transformations*, J. Stat. Phys. 19 (1978), no. 1, 25–52.

[15] D. P. Feldman, *Chaos and Fractals, An Elementary Introduction*, Oxford University Press 2012.

[16] J. Graczyk, G. Światek, *Induced expansion for quadratic polynomials*, Ann. Sci Éc. Norm. Súp. 29 (1996) 399–482.

[17] J. Guckenheimer, *On the Bifurcation of Maps of the Interval*, Inventiones mathematicae 39 (1977) 165–178.

[18] J. Guckenheimer, *Sensitive dependence on initial conditions for unimodal map*, Commun. Math. Phys. 70 (1979) 133–160.

[19] T. Hall (2010) Kneading theory. Scholarpedia, 5(11):3956.

[20] M. V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys. 81 (1981) 39–88.

[21] D. Lind, B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press New York, NY, USA 1995.

[22] M. Lyubich, *Combinatorics, geometry and attractors of quasi-quadratic maps*, Ann. of Math. 140 (1994) 347-–404 and Erratum Manuscript (2000).

[23] W. de Melo, *Rigidity and renormalization in one-dimensional dynamical systems*, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math (1998), Extra Vol. II, 765–778.

[24] W. de Melo, S. van Strien, *One-dimensional dynamics*, Springer Verlag 1993.

[25] M. Metropolis, M. L. Stein and P. R. Stein, *On finite limit sets for transformations of the interval*, J. Combin. Theory A 15 (1973), 25–44.

[26] J. Milnor and W. Thurston, *On iterated maps of the interval: I, II*, Preprint 1977., Published in Lect. Notes in Math. 1342, Springer, Berlin New York (1988) 465–563.

[27] M. Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, World Scientific Publishing 2000.

[28] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. 67 (1980), no. 1, 45–63.

[29] L. Rempe-Gillen, S. van Strien, *Density of hyperbolicity for classes of real transcendental entire functions and circle maps*, Duke Math. J. 164 (2015), no. 6, 1079–1137.

[30] D. Singer, *Stable orbits and bifurcation of maps of the interval*, SIAM J. Appl. Math. 35 (1978), no.2, 260–267.

[31] A. Sharkovsky, *Coexistence of cycles of a continuous map of the line into itself*, (Russian) Ukrain. Math. Zh. 16 (1964), 61–71.

[32] S. Silverman, *On maps with dense orbits and the definition of chaos*, Rocky Mountain Jour. Math. 22 (1992), 353–375.

[33] S. van Strien, *Smooth dynamics on the interval*, New Directions in Dynamical Systems, Cambridge Univ. Press, (1987) 57–119.

[34] M. Tsujii, *A simple proof for monotonicity of entropy in the quadratic family*, Ergod. Th. Dynam. Sys. 20 (2000), 925–933.