Problem Sheet I

Introduction to Random Matrices, VDI Summer School, Weißensee
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1 Differences in level repulsion for real symmetric and complex Hermitian matrix ensembles

The goal of this exercise is to demonstrate that the level repulsion asymptotics can depend on the symmetry class of the matrix ensemble, i.e., whether the matrices are real symmetric or complex Hermitian.

Problem 1. *Let*

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$$

be a 2×2 real symmetric or complex Hermitian random matrix with independent continuously distributed entries. Denote the real eigenvalues of H by λ_1, λ_2.

(i) If H is real symmetric, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^2$$

in the sense that the probability scales like ϵ^2 for small ϵ.

(ii) If H is complex Hermitian and the real and imaginary part of h_{12} are independent and continuously distributed, show that

$$\mathbf{P}(|\lambda_1 - \lambda_2| \leq \epsilon) \sim \epsilon^3.$$

□

2 Wigner semicircle law

The goal of this exercise is to prove the Wigner semicircle law. Assume that $H = (h_{ij})_{i,j=1}^N$ is a matrix of mean zero $\mathbf{E}\, h_{ij} = 0$ random variables such that the h_{ij} are independent for $i \leq j$ and $h_{ij} = \overline{h_{ji}}$ otherwise. Moreover assume that $\mathbf{E}\,|h_{ij}|^2 = 1/N$, and for simplicity also that $\mathbf{E}\,|h_{ij}|^k \leq C_k N^{-k/2}$ for some constants C_k. Denote the eigenvalues of H by $\lambda_1, \dots, \lambda_N \in \mathbb{R}$ and define the spectral measure of H as

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}.$$

Wigner's semicircle law states that under the assumptions above we have that almost surely

$$\lim_{N\to\infty} \int f(x)\,\mathrm{d}\mu_N(x) = \int_{-2}^{2} f(x) \frac{\sqrt{4-x^2}}{2\pi}\,\mathrm{d}x \tag{1}$$

for each continuous bounded function $f\colon [-10, 10] \to \mathbb{C}$.

Problem 2 (Moments of H)**.** *We first consider the k-th moments of μ_N. For $k \geq 0$ we compute*

$$m_{k,N} := \int x^k\,\mathrm{d}\mu_N(x) = \frac{1}{N}\sum_{i=1}^N \lambda_i^k = \frac{1}{N}\operatorname{Tr} H^k = \frac{1}{N}\sum_{i_1,\dots,i_k=1}^N h_{i_1 i_2} h_{i_2 i_3} \dots h_{i_k i_1}. \tag{2}$$

The summation in (2) *is over all cycles of the complete graph on the vertex set $[N] := \{1, \dots, N\}$.*

(i) *Use the independence assumption to show that for the computation of* $\mathbf{E}\, m_{k,N}$ *only those cycles contribute where each undirected edge is traversed at least twice. Conclude that the number of vertices in the contributing cycles is at most* $\lfloor k/2 \rfloor + 1$.

(ii) *Conclude from* (i) *that for odd* k, $\mathbf{E}\, m_{k,N} = \mathcal{O}\left(N^{-1/2}\right)$, *while for even* k,

$$\mathbf{E}\, m_{k,N} = N^{-k/2-1} \left| \left\{ \text{Cycles in } [N] \text{ with } k/2+1 \text{ vertices traversing each edge exactly twice} \right\} \right| + \mathcal{O}\left(N^{-1/2}\right).$$

(iii) *Define a Dyck path of length* $2k$ *as a path from* $(0,0)$ *to* $(2k,0)$ *in* $[2k]\times\mathbb{Z}_+$ *such that the vertex succeeding* (i,j) *is either* $(i+1,j+1)$ *or* $(i+1,j-1)$. *Define* C_k *to be the the number of such Dyck paths of length* $2k$. *Show that the number of cycles of length* $2k$ *in* $[N]$ *with* $k+1$ *unique vertices traversing each edge exactly twice is* $C_k N(N-1)\dots(N-k) = C_k N^{1+k} + \mathcal{O}\left(N^k\right)$.

(iv) *Conclude that*

$$\mathbf{E}\, m_{k,N} = \begin{cases} C_{k/2} & k \text{ even}, \\ 0 & k \text{ odd} \end{cases} + \mathcal{O}\left(N^{-1/2}\right).$$

(v) *[Optional] Prove that* $\mathbf{Var}\, m_{k,N} = \mathcal{O}\left(N^{-2}\right)$ *and conclude that* $m_{k,N}$ *converges almost surely to* $C_{k/2}$ *for even* k *and* 0 *for odd* k.

□

From Problem 2 there are multiple ways of concluding (1). One option would be to realize that both the $2k$-th moments of the semicircular distribution and the number of Dyck paths C_k are given by Catalan numbers $(2k)!/k!(k+1)!$. The integration, however, is somewhat tedious and we therefore follow an alternative path which also does not require knowing the semicircular distribution in advance.

Problem 3 (Stieltjes transform from moments)**.**

(i) *Prove the Recursion relation*

$$C_k = C_0 C_{k-1} + C_1 C_{k-2} + \dots + C_{k-1} C_0 \tag{3}$$

for $k \geq 1$.

(ii) *Let* μ *be the probability distribution*[1] *whose* k*-th moments are given by* $C_{k/2}$ *for even* k *and* 0 *for odd* k. *Denote the Stieltjes transform of* μ *by*

$$m(z) = \int \frac{\mathrm{d}\mu(x)}{x-z} = -\frac{1}{z}\int \frac{\mathrm{d}\mu(x)}{1-x/z} = -\sum_{k\geq 0} \frac{1}{z^{1+k}} \int x^k \,\mathrm{d}\mu(x) = -\sum_{k\geq 0} \frac{C_k}{z^{2k+1}},$$

(3) *where the geometric series expansion holds for large enough* z. *Conclude from* (3) *that* $m(z)$ *satisfies the functional equation*

$$m(z) = -\frac{1}{z} - \frac{1}{z} m(z)^2.$$

(iii) *By solving the quadratic equation conclude that* m *is the Stieltjes transform of the semicircular distribution.*

□

[1] Uniqueness of such a distribution follows e.g. from Carleman's condition