Problem Sheet II

Introduction to Random Matrices, VDI Summer School, Weißensee
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1 Cumulant expansion

Problem 1. *Show that for a zero mean Gaussian random variable x and a smooth function $f\colon \mathbb{R}\to\mathbb{R}$ we have* $\mathbf{E}\, x f(x) = (\mathbf{E}\, x^2)(\mathbf{E}\, f'(x))$.
Hint. *Integration by parts.*

□

We now generalise the expansion from Problem 1 to non-Gaussian multivariate random vectors. Let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ be a random vector. The characteristic function $\mathbb{R}^n\ni\omega\mapsto \mathbf{E}\, e^{\mathrm{i}\omega\cdot x}$ has the mixed moments of x as its Taylor coefficients, i.e.

$$\mathbf{E}\, e^{i\omega\cdot x} = \sum_{k=0}^{N}\sum_{j_1,\dots,j_k=1}^{n} (\mathbf{E}\, x_{i_1}\dots x_{i_k})\frac{(\mathrm{i}\omega_{j_1})\dots(\mathrm{i}\omega_{j_k})}{k!}. \tag{1}$$

Conversely, the Taylor coefficients of the log-characteristic function $\omega\mapsto\log\mathbf{E}\, e^{\mathrm{i}\omega\cdot x}$

$$\log\mathbf{E}\, e^{\mathrm{i}\omega\cdot x} = \sum_{k=1}^{\infty}\sum_{j_1,\dots,j_k=1}^{n}\kappa(x_{j_1},\dots,x_{j_k})\frac{(\mathrm{i}\omega_{j_1})\dots(\mathrm{i}\omega_{j_k})}{k!}, \tag{2}$$

are commonly called (mixed) *cumulants* $\kappa(x_{i_1},\dots,x_{i_k})$. Cumulants are particularly useful for computing expectations of the form $\mathbf{E}\, x_i f(x)$ for general smooth functions f.

Problem 2. *Prove that for smooth functions f we at least formally have*

$$\mathbf{E}\, x_i f(x) = \sum_{k=0}^{\infty}\sum_{j_1,\dots,j_k=1}^{n}\kappa(x_i,x_{j_1},\dots,x_{j_k})\frac{\mathbf{E}\,\partial_{i_1}\dots\partial_{i_k} f(x)}{k!}. \tag{3}$$

□

In the case of Gaussian random vectors, the expansion (3) takes a particularly simple form. Indeed, the characteristic function of a Gaussian vector x with mean $\mu=(\mu_i)$ and covariance $\Sigma=(\Sigma_{ij})$, $\Sigma_{ij}=\mathbf{Cov}\,(x_i,x_j)=(\mathbf{E}\, x_i x_j)-(\mathbf{E}\, x_j)(\mathbf{E}\, x_j)$, is simply given by

$$\mathbf{E}\, e^{\mathrm{i}\omega\cdot x} = \exp\Big(\mathrm{i}\omega^t\mu - \frac{1}{2}\omega^t\Sigma\omega\Big).$$

Therefore, the log characteristic function reads

$$\log\mathbf{E}\, e^{\mathrm{i}\omega\cdot x} = \sum_i \mu_i(\mathrm{i}\omega_i) + \sum_{i,j}\Sigma_{ij}\frac{(\mathrm{i}\omega_i)(\mathrm{i}\omega_j)}{2!},$$

from which we can infer that cumulants of order 3 and higher vanish, and that

$$\kappa(x_i)=\mathbf{E}\, x_j,\qquad \kappa(x_i,x_j)=\mathbf{Cov}\,(x_i,x_k)=\mathbf{E}\, x_i x_j-(\mathbf{E}\, x_i)(\mathbf{E}\, x_j). \tag{4}$$

In particular, (3) includes the well known integration by parts formula

$$\mathbf{E}\, x_i f(x) = \sum_j (\mathbf{E}\, x_i x_j)\,\mathbf{E}\,\partial_j f(x) \tag{5}$$

for zero mean $\mathbf{E}\, x=0$ Gaussian random vectors x.

The relations between moments and cumulants from (4) extend beyond Gaussian random variables. By expressing cumulants purely in terms of mixed moments we actually can provide an alternative purely combinatorial definition (6) of cumulants.

Problem 3. *[Optional] For a partition $\mathcal{P} \vdash [n]$ of $[n] := \{1, \ldots, n\}$, we introduce the notations*

$$\kappa^{\mathcal{P}} := \prod_{A \in \mathcal{P}} \kappa(x_i \mid i \in A), \qquad M^{\mathcal{P}} := \prod_{A \in \mathcal{P}} \mathbf{E} \prod_{i \in A} x_i$$

for products of cumulants and moments. If a partition $\mathcal{Q}$ is finer *than $\mathcal{P}$ we write $\mathcal{Q} \leq \mathcal{P}$.*

(i) Use (1)–(2) *to show that for any partition $\mathcal{P}$ we have*

$$M^{\mathcal{P}} = \sum_{\mathcal{Q} \leq \mathcal{P}} \kappa^{\mathcal{Q}}.$$

(ii) Use the general Möbius inversion on the incidence algebra of the partially ordered set of partitions $\mathcal{P} \vdash [n]$ to show that

$$\kappa^{\mathcal{P}} = \sum_{\mathcal{Q} \leq \mathcal{P}} \mu(\mathcal{Q}, \mathcal{P}) M^{\mathcal{Q}}, \qquad \mu(\mathcal{Q}, \mathcal{P}) = (-1)^{|\mathcal{Q}| - |\mathcal{P}|} 0!^{r_1} 1!^{r_2} \ldots (|\mathcal{Q}| - 1)!^{r_{|\mathcal{Q}|}},$$

where r_i is the number of blocks of $\mathcal{P}$ which contain exactly i blocks of $\mathcal{Q}$. In particular, conclude that

$$\kappa(x_1, \ldots, x_n) = \sum_{\mathcal{Q} \vdash \{1, \ldots, n\}} (-1)^{|\mathcal{Q}| - 1} (|\mathcal{Q}| - 1)! \prod_{A \in \mathcal{Q}} \mathbf{E} \prod_{i \in A} x_i, \tag{6}$$

□

2 Smallness of the error term in the local law

Problem 4. *Consider a general Hermitian random matrix $H = H^*$ with mean $A = \mathbf{E} H$, $H = A + W$.*

(i) Show that for (possibly correlated) Gaussian W we have

$$1 = \mathbf{E}(A - \mathcal{S}[G] - z)G, \qquad \mathcal{S}[R] := \mathbf{E} WRW = \sum_{a,b,c,d} \kappa(w_{ab}, w_{cd}) \Delta^{ab} R \Delta^{cd}, \tag{7}$$

where Δ^{ab} is a matrix of 0's with an 1 in the (a, b)-entry.

(ii) Convince yourself that in the case of Wigner-type matrices with independent entries (up to symmetry) with $S = (s_{ij}), T = (t_{ij})$, $s_{ij} := \mathbf{E} |w_{ij}|^2$, $t_{ij} := \mathbf{E} w_{ij}^2$ we have

$$\mathcal{S}[R] = \operatorname{diag}(S \operatorname{diag} R) + T \odot R^t,$$

where $\odot$ denotes the entrywise (Hadamard) product. In particular in the case of the Gaussian unitary ensemble with $s_{ij} = 1/N$, $t_{ij} = 0$ we have $\mathcal{S}[R] = \langle R \rangle := N^{-1} \operatorname{Tr} R$.

□

From (7) it is reasonable to expect that the solution M to the deterministic equation

$$1 = (A - \mathcal{S}[M] - z)M, \quad M = M(z), \quad \Im M = \frac{M - M^*}{2\mathrm{i}}, \quad \Im z > 0$$

is a good approximation of the resolvent G. We call the error term in this approximation

$$D := 1 - (A - \mathcal{S}[G] - z)G = WG + \mathcal{S}[G]G.$$

It remains to show that the error term D is indeed small as long as $\eta \gg N^{-1}$, i.e. above the typical eigenvalue spacing in the bulk. We already saw in (7) that $\mathbf{E} D = 0$. The goal of the next exercise is to show that $\mathbf{E} |\langle D \rangle|^2$ is also small.

Problem 5. *Assume that if W is a zero mean $\mathbf{E} W = 0$ (possibly correlated) Gaussian random matrix.*

(i) Assume that the entries of W are correlated in such a way that the operator $\kappa = (\kappa_{ab,cd})$, $\kappa_{ab,cd} := \kappa(w_{ab}, w_{cd})$ viewed as a $N^2 \times N^2$-matrix is bounded in operator norm $\|\kappa\| \leq C/N$ for some constant C. Then use (5) *to prove*

$$\mathbf{E} |\langle D \rangle|^2 \lesssim \frac{1}{N^3} \sum_{ab} \mathbf{E} |G_{ab}|^2 + \frac{1}{N^4} \sum_{abcd} \mathbf{E} |G_{ab}|^2 |G_{cd}|^2.$$

(ii) Prove the so called Ward identity

$$\sum_a |G_{ab}|^2 = (G^* G)_{bb} = \frac{(\Im G)_{bb}}{\eta}.$$

(iii) Conclude that

$$\mathbf{E}\,|\langle D\rangle|^2 \lesssim \mathbf{E}\,\frac{\langle \Im G\rangle}{N^2\eta} + \mathbf{E}\,\frac{\langle \Im G\rangle^2}{(N\eta)^2}.$$

Note that the operator norm of κ in the independent case can be bounded by

$$\|\kappa\| \le \max_{ab} \sum_{cd} |\kappa_{ab,cd}| = \max_{ab}\Big(\mathbf{E}\,|w_{ab}|^2 + \big|\mathbf{E}\,w_{ab}^2\big|\Big),$$

which is usually assumed to be bounded by C/N. □